DOCUMENT RESUME

ED 339 718 TM 017 553

AUTHOR Tatsuoka, Kikumi K.

TITLE Boolean Algebra Applied to Determination of Universal Set of Knowledge States.

INSTITUTION Educational Testing Service, Princeton, N.J.

SPONS AGENCY Office of Naval Research, Arlington, VA. Cognitive and Neural Sciences Div.
REPORT NO RR-91-44-ONR
PUB DATE Aug 91
CONTRACT ONR-N00014-90-J-1307
NOTE 46p.
PUB TYPE Reports - Evaluative/Feasibility (142)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Algorithms; *Classification; Cognitive Processes; Equations (Mathematics); *Error Patterns; *Item Response Theory; *Knowledge Level; *Mathematical Models; Misconceptions; *Student Reaction
IDENTIFIERS Boolean Algebra

ABSTRACT

Diagnosing cognitive errors possessed by examinees can be considered as a pattern classification problem that is designed to classify a sequential input of stimuli into one of several predetermined groups. The sequential inputs in this paper's context are item responses, and the predetermined groups are various states of knowledge resulting from misconceptions or different degrees of incomplete knowledge in a domain. In this study, the foundations of a combinatorial algorithm that will provide the universal set of states of knowledge will be introduced. Each state of knowledge is represented by a list of "can/cannot" cognitive tasks and processes (cognitively relevant attributes or latent variables) that are usually unobservable. A Boolean descriptive function is introduced as a mapping between the attribute space spanned by latent attribute variables and the item response space spanned by the item score variables. This function uncovers the unobservable content of a "black box." Once all possible classes are retrieved explicitly and expressed by a set of ideal item response patterns described by a "can/cannot" list of latent attributes, the notion of bug distributions and statistical pattern classification techniques will enable the accurate diagnosis of students' states of knowledge. Moreover, investigations on algebraic properties of these logically-derived-ideal-response patterns will provide insight into the structures of the test and dataset. There are 11 references and three illustrative tables. (Author/SLD)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

TM

RR-91-44-ONR

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
☑ This document has been reproduced as received from the person or organization originating it.
☐ Minor changes have been made to improve reproduction quality.
• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

ED339718

BOOLEAN ALGEBRA APPLIED TO DETERMINATION OF UNIVERSAL SET OF KNOWLEDGE STATES

Kikumi K.Tatsuoka

This research was sponsored in part by the
Cognitive Science Program
Cognitive and Neural Sciences Division
Office of Naval Research, under
Contract No. N00014-90-J-1307
R&T 4421559

Kikumi K. Tatsuoka, Principal Investigator

Educational Testing Service
Princeton, New Jersey

August 1991

Reproduction in whole or in part is permitted
for any purpose of the United States Government.

Approved for public release; distribution unlimited.

TM017553

BEST COPY AVAILABLE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): RR-91-44-ONR	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: Educational Testing Service; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Cognitive Science Program, Office of Naval Research (Code 1142CS)
6c. ADDRESS (City, State, and ZIP Code): Princeton, NJ 08541	7b. ADDRESS (City, State, and ZIP Code): 800 N. Quincy Street, Arlington, VA 22217-5000
8a. NAME OF FUNDING/SPONSORING ORGANIZATION; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: N00014-90-J-1307
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. 61153 N; PROJECT NO. RR 04204; TASK NO. RR04204-01; WORK UNIT ACCESSION NO. R&T4421559

11. TITLE (Include Security Classification): Boolean algebra applied to determination of universal set of knowledge states (Unclass.)

12. PERSONAL AUTHOR(S): Kikumi K. Tatsuoka

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM 1989 TO 1992	August 1991	37

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Knowledge states; classification; diagnosing misconceptions; Boolean algebra.
05	10		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Diagnosing cognitive errors possessed by examinees can be considered as a pattern classification problem which is designed to classify a sequential input of stimuli into one of several predetermined groups. The sequential inputs in our context are item responses and the predetermined groups are various states of knowledge resulting from misconceptions or different degrees of

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Dr. Susan Chipman	22b. TELEPHONE (Include Area Code): 703-696-4318; 22c. OFFICE SYMBOL: ONR 1142CS

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): RR-91-44-ONR	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: Educational Testing Service; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Cognitive Science Program, Office of Naval Research (Code 1142CS)
6c. ADDRESS (City, State, and ZIP Code): Princeton, NJ 08541	7b. ADDRESS (City, State, and ZIP Code): 800 N. Quincy Street, Arlington, VA 22217-5000
8a. NAME OF FUNDING / SPONSORING ORGANIZATION; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: N00014-90-J-1307
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61153 N	RR 04204	RR04204-01	R&T4421559

11. TITLE (Include Security Classification): Boolean algebra applied to determination of universal set of knowledge states (Unclass.)

12. PERSONAL AUTHOR(S): Kikumi K. Tatsuoka

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Technical	FROM 1989 TO 1992	August 1991	37

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Knowledge states; classification; diagnosing misconceptions; Boolean algebra.
05	10		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Diagnosing cognitive errors possessed by examinees can be considered as a pattern classification problem which is designed to classify a sequential input of stimuli into one of several predetermined groups. The sequential inputs in our context are item responses and the predetermined groups are various states of knowledge resulting from misconceptions or different degrees of

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL
Dr. Susan Chipman	703-696-4318	ONR 1142CS

DD Form 1473, JUN 86 — Previous editions are obsolete. S/N 0102-LF-014-6603

SECURITY CLASSIFICATION OF THIS PAGE: Unclassified

Boolean Algebra Applied to Determination of Universal Set of Knowledge States

Kikumi K. Tatsuoka

August 1991

Copyright © 1991. Educational Testing Service. All rights reserved.

ABSTRACT

Diagnosing cognitive errors possessed by examinees can be considered as a pattern classification problem which is designed to classify a sequential input of stimuli into one of several predetermined groups. The sequential inputs in our context are item responses and the predetermined groups are various states of knowledge resulting from misconceptions or different degrees of incomplete knowledge in a domain. In this study, the foundations of a combinatorial algorithm that will provide the universal set of states of knowledge will be introduced. Each state of knowledge is represented by a list of "can/cannot" cognitive tasks and processes (called cognitively relevant attributes or latent variables) which are usually unobservable. A Boolean descriptive function will be introduced as a mapping between the attribute space spanned by latent attribute variables and the item response space spanned by item score variables. The Boolean descriptive function plays the role of uncovering the unobservable content of a black box. Once all the possible classes are retrieved explicitly and expressed by a set of ideal item response patterns which are described by a "can/cannot" list of latent attributes, the notion of bug distributions and statistical pattern classification techniques will enable us to diagnose students' states of knowledge accurately. Moreover, investigations on algebraic properties of these logically-derived-ideal-response patterns will provide an insight into the structures of the test and dataset.

Introduction

A typical pattern classification problem is to classify a sequential input of stimuli into one of several predetermined groups. The predetermined groups are considered, in our context, as latent classes which represent various states of knowledge and capabilities, and the stimuli are item response patterns. Tatsuoka (1983, 1985) introduced a cognitive error diagnostic model (called rule space) in which a student's response pattern to the items is classified into one of the predetermined latent classes. Each latent class consists of binary patterns that deviate from a given ideal response pattern by various numbers of slippages. Tatsuoka & Tatsuoka (1987) introduced the slippage probabilities and showed that such a class of response patterns follows a statistical distribution (called a Bug distribution). The ideal response pattern is the outcome of the perfectly consistent execution of some erroneous rule of operation or the response pattern corresponding to some state of knowledge and capabilities without errors of measurement. An error analysis or a task anlysis usually provides a list of erroneous rules of operations and/or various sources of misconceptions which are regarded as latent classes in this paper. However, it is important to have a systematic method for obtaining an appropriate list of ideal response patterns automatically. The method must be

applicable to any domain of interest. In this paper, such a method and the theoretical foundation of the method are introduced. The theoretical foundation is built upon algebraic relations between observable item patterns and latent score patterns of various cognitive tasks. Boolean Lattice theory is applied to develop the theoretical foundation of a test and data structure.

An Incidence Matrix and Binary Scoring

Suppose that the underlying characteristics of a domain of interest are well identified and involvement relationships between the latent attribute variables A_k, k=1,...,K (also called cognitively relevant attributes) and items are coded by a binary matrix. The matrix is called an incidence matrix. Let the incidence matrix be a K x n matrix Q where K is the number of attributes and n is the number of items. The row vectors of Q, A_k, k = 1,..., K indicate which items involve the attribute A_k. Let latent variable Y_k be the score of attribute task A_k; that is, Y_k = 1 if attribute A_k is correctly performed and Y_k = 0, otherwise (if A_k is not a task, and the word "score" is not suitable, then Y_k = 1 could signify "applicable", "belonging to" or any "affirmative adjective"). Let X_j be a score variable of item j and assume that X_j takes the value 1 for the correct answer

and 0 for wrong answers. The relationship between latent-score pattern $y = (Y_1, \ldots Y_K)$ and the observable item score X_j is given by Equation (1):

$$X_j = \prod_{k=1}^{K} Y_k^{Q_{kj}} \qquad j=1, \ldots, n \tag{1}$$

This equation implies that a response to item j will be correct if and only if latent scores Y_k of attribute A_k for $Q_{kj} = 1$ are <u>all</u> equal to 1. If any one of such latent scores is zero, then the item score X_j becomes zero. Needless to say, the meaning of $Q_{kj} = 0$ and $Y_k = 0$ should not be confused because Q_{kj} is an involvement index of attribute A_k to item j while Y_k is the score of attribute task A_k. The latent score pattern for item j shall be expressed by

$$z_j = (Y_1Q_{1j}, Y_2Q_{2j}, \ldots, Y_kQ_{kj}) \tag{2}$$

where $Z_{kj} = Y_kQ_{kj}$ does not exist when $Q_{kj} = 0$.

Further let us assume the conditional independence of latent variables Y_k ($k = 1, \ldots, K$) and manifest variables X_j ($j = 1, \ldots, n$) for each performance level θ. Let t be the total score of a latent score pattern y where we assume a special case, $Q_{kj} = 1$ for

$k = 1, \ldots, K$, and p_k be the probability of attribute A_k to be performed correctly,

$$t = \sum_{k=1}^{K} Y_k \tag{3}$$

Then the random variable t follows a binomial distribution if the attribute probabilities p_k are the same for all k, and a compound binomial distribution if the probabilities are different.

$$\text{Prob}(t|\theta) = \left\{ \sum_{m=0}^{t} \left\{ \sum_{\Sigma Y_k = m} \prod_{k=1}^{K} p_k^{Y_k} (1-p_k)^{1-Y_k} \right\} \right\} \tag{4}$$

The probability of getting the total score of K, or equivalently the pattern of all ones (1, 1,..., 1) is given by the last term of equation (4).

$$\text{Prob}(t = K|\theta) = \prod p_k \tag{5}$$

Let s_j be the total score of a latent pattern z_j for item j, then the relationships parallel to equations (4) and (5) for the variables s_j, z_j and the probabilities p_k are given by equations (6) and (7), respectively.

$$\text{Prob}(s_j|\theta) = \left\{ \sum_{m=0}^{s_j} \left\{ \sum_{\Sigma Z_{jk} = s_j} \prod_{k=1}^{K} p_k^{Z_{jk}} (1-p_k)^{1-Z_{jk}} \right\} \right\} \qquad (6)$$

The probability of getting a particular pattern $z_j = 1$ is given by equation (7),

$$\text{Prob}\left[s_j = \sum_{k=1}^{K} Q_{kj} | \theta \right] = \prod_{k=1}^{K} p_k \qquad (7)$$

When item score X_j is not binary and the response to item j is scored by taking some partial knowledge into account, then the above discussion needs to be modified.

An Incidence Matrix And Partial Credit Scoring

The elements of a latent pattern $z_j = (Z_{1j}, Z_{2j}, \ldots, Z_{Kj})$ of item j can be replaced by integers or real numbers. Each element Z_{kj} can be the number of attributes which an examinee answered correctly or the weighted sum of the number of attributes answered correctly. That is:

$$X_j = \sum_{k \in Q_{kj} = 1} W_{kj} Z_{kj}, \qquad (8)$$

where $Z_{kj} = 1$ if and only if attribute A_k is involved in item j and an examinee performed A_k correctly.

When W_{kj} is equal to 1 for k = 1,..., K, then X_j becomes simply the number of correct attributes. The larger the X_j value is, the higher the level of performance is. Thus, graded response or partial credit models (Samejima,F., 1969; Masters,G.N.,1982) can be applied. However, Z_{kj}, $k\epsilon(Q_{kj} - 1)$ are usually not observable. If a multiple-choice item is constructed so as to have various subsets of scores of Z_{kj} for the alternatives, then it is possible to apply graded, partial credit or Polychotomous models. The partial credit model is formulated for situations in which ordered response choices are free to vary in number and difficulty from item to item. The restriction of the model is that tests are constructed with an ordered response format. Polychotomous models (Bock,R.D.,1972) do not require the ordered response format and are applicable to multinomial response categories.

When the weights are not 1, then the W_{kj}'s indicate that the quality of A_k varies over the attributes. Some attributes are more difficult while others are less so. It is well known that there are $\binom{s}{sj}$ ways to get the total score of s_j from s different attributes. Some combinations are cognitively more important than

others. It will provide us with useful information for constructing a good item pool for constructed response items or selection of distractors in multiple choice items.

Lattice and Boolean Algebra

In the previous section, the attributes were introduced as the row vectors of the incidence matrix Q and denoted by vectors A_k, k=1,..., K. For example, let us consider the 3 x 5 incidence matrix shown below, where $i_1,\ldots, i_5$ are items and A_1, A_2, and A_3 are attributes:

$$Q = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccccc} i_1 & i_2 & i_3 & i_4 & i_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad (9)$$

There are three row vectors, A_1 = (1 0 1 0 1), A_2 = (0 1 0 0 1) and A_3 = (0 1 1 1 1). In other words, attribute A_1 is involved in items 1, 3 and 5, attribute A_2 is in items 2 and 5 and attribute A_3 is in 2, 3, 4 and 5. Therefore, the attributes can also be expressed by a set theoretical notation like A_1 = {1, 3, 5}, A_2 = {2, 5} and A_3 = {2, 3, 4, 5}. When we discuss the attributes in the context of set theory, the attributes are written in non-boldface capital letters as $A_1,.., A_K$. If an incidence matrix Q happens to be the identity matrix of order K = n, then A_k contains

a single item, and $A_k = (k)$.

Let L be a set of subsets obtained from the set of K numbers, $J = (1, 2, \ldots, K)$. L will be a lattice and Boolean algebra. Lattice and Boolean algebra have been discussed in the field of abstract algebra and they have many interesting properties. They have been applied to digital computer systems and proved to be very useful in providing a simple and precise foundation for the analysis of combinatorial switching circuits. These properties will play a crucial role in achieving our goal which is to obtain the universal set of ideal response patterns (or all the possible states of knowledge and capabilities) obtainable from a given incidence matrix. Let us start from the definition of a lattice.

Definition 1 A set of sets L is said to be a lattice if two binary compositions $\cup$ and $\cap$ are defined on its subsets (called elements hereafter) and they satisfy the following relations:

l_1 $A \cup B = B \cup A$, $A \cap B = B \cap A$

l_2 $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$

l_3 $A \cup A = A$, $A \cap A = A$

l_4 $(A \cup B) \cap A = A$, $(A \cap B) \cup A = A$

The above conditions are equivalent to saying that a lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. The l.u.b. and g.l.b. of any

elements A and B in L are given by the union and intersection, $A \cup B$, and $A \cap B$, respectively. Similarly, $(A \cup B) \cup C$ is the l.u.b. of A, B, C and $(A \cap B) \cap C$ is the g.l.b. The order $\geq$ in L is defined by Definition 2:

Definition 2 For any pair of elements A and B in L, $A \geq B$ if and only if $A \cup B = A$ or $A \cap B = B$.

Definition 2 provides us with an equivalent condition for L to be a lattice. This order satisfies the asymmetric (if $A \geq B$ and $B \geq A$ then $A = B$) and transitivity laws (if $A \geq B$ and $B \geq C$ then $A \geq C$), thus L becomes a partially ordered set. Let us further define I and 0 as follows:

$$I = \bigcup_{k=1}^{K} A_k \quad \text{and} \quad 0 = \bigcap_{k=1}^{K} A_k \tag{10}$$

then I and 0 belong to L. If the distributive law,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{11}$$

is satisfied, then L is called a modular lattice. The modular condition has an alternative definition: if $A \geq B$ and $A \cup C = B \cup C$ and $A \cap C = B \cap C$ for any C in L, then $A = B$.

The third important operation is complementation.

Definition 3 The complement A' of A is defined by $A' \cup A = 1$ and

$A' \cap A = 0$.

For example, the lattice of a set of subsets is complemented if the complement of a subset A is the usual set-theoretic complement--that is, the elements of J that do not belong to A.

Definition 4 A Boolean algebra is a lattice with 1 and 0, the distributive law and complementation.

Definition 4 implies that our lattice L is also a Boolean algebra. The most important elementary properties of complements in a Boolean algebra may be stated as follows:

Theorem 1 The complement A' of any element A of a Boolean Algebra L is uniquely determined. The mapping $A \to A'$ is one to one, onto itself. Then the mapping satisfies conditions 1 and 2:

1. $(A')' = A$
2. $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

The proof may be found in Birkoff (1970).

A Boolean algebra becomes a Ring with the two operations + and x where + is the union set of A and B and x is the intersection of A and B.

Definition 5 For A and B in L, the addition + of A and B is defined by $A + B = A \cup B$ and the product x is defined by $A \times B = A \cap B$. Thus L becomes a Ring.

It is obvious that L satisfies commutative laws, associative laws, the identity laws $A + 0 = A$, $A \times I = A$, and Idempotent law

$A + A = A$ with respect to the new operations + and x.

Distributive law is also satisfied. In summary,

1. $A + B = B + A$, $A \times B = B \times A$

Commutative Laws

2. $(A + B)' = A' \times B'$, $(A \times B)' = A' + B'$

Complementation

3. $(A + B) + C = A + (B + C)$, $(A \times B) \times C = A \times (B \times C)$

Associative Laws

4. $A + O = O + A = A$, $A \times I = I \times A = A$

Identity

5. $A + A = A$, $A \times A = A$

Idempotence

6. $(A + B) \times C = A \times C + B \times C$

Distributive Law

The relationship between the attribute vectors A_k ($k = 1, \ldots, K$) and the Ring L just introduced will be clarified.

Attribute Response Space and Item Response Space

When an incidence matrix is the identity matrix of order K, then A_k will be the unit vector $e_k = (0, .., 1, 0, .., 0)$, whose k-th element is 1 and the other elements are zero. A Boolean

lattice L will then consist of a set of attributes where attribute A_k corresponds one-to-one to item k or equivalently to e_k. Therefore, L can be considered as a set of sets of items, or equivalently as a set of sets of e_ks. In order to distinguish between these two sets, the set of sets of attributes is denoted by the same notation, L and the set of sets of items (or sets of e_k) is denoted by RL, in other words Boolean Algebra of Item Response Patterns. Both L and RL are K-dimensional spaces since the incidence matrix is the identity of order K. If an incidence matrix is not the identity then RL_1 which associates with a non-identity incidence matrix becomes a subspace of RL. It is very difficult, in practice, to construct an item-pool whose incidence matrix is the identity. Each item in the identity-incidence matrix must contain one and only one attribute. It is very common that an item involves several attributes and two different items usually involve two different sets of attributes. In practice most incidence matrices are usually more complicated than the identity matrix and their columns and rows contain several ones in a variety of cells.

In the earlier example of 3 x 5 matrix, A_1 = (1 0 1 0 1) corresponds to set A_1 = (1, 3, 5); A_2 = (0 1 0 0 1) to A_2 = (2, 5); and A_3 = (0 1 1 1 1) to A_3 = (2, 3, 4, 5). The union set of A_1 and A_2, (1, 2, 3, 5) corresponds to the addition of

$A_1 + A_2 = (1\ 1\ 1\ 0\ 1)$ in terms of elementwise Boolean addition. Boolean addition is defined by $1 + 1 = 1$, $1 + 0 = 1$, $0 + 1 = 1$ and $0 + 0 = 0$. The intersection of A_1 and A_2, (5) corresponds to the product of $A_1 \times A_2 = (0\ 0\ 0\ 0\ 1)$ in terms of elementwise Boolean multiplication of 0 and 1. Boolean multiplication follows the rules, $0 \times 0 = 0$, $1 \times 0 = 0 \times 1 = 0$ and $1 \times 1 = 1$. It is clear that these operations satisfy the above relations 1 through 6. The complement of A_k is A'_k whose elements are obtained by switching each element of A_k to the opposite; thus complement of A_1 is $(0\ 1\ 0\ 1\ 0)$, A'_2 is $(1\ 0\ 1\ 1\ 0)$ and A'_3 is $(1\ 0\ 0\ 0\ 0)$. It is also clear that $A_k + A'_k$ is equal to 1.

Suppose A_k, $k = 1, \ldots, K$ are the row vectors of such a general incidence matrix, and let RL_1 be a set of sets of the attribute vectors. Then RL_1 becomes a sublattice of RL which is derived from the set of all the response patterns. A subset RL_1 of RL is called a sublattice if it is closed with respect to the binary compositions $\cap$ and $\cup$. Further Theorem 2 shows that RL_1 becomes a subring of RL also.

Theorem 2 A set RL_1 of sets of row vectors of an incidence matrix Q is a Boolean algebra with respect to elementwise Boolean addition and multiplication of 0 and 1.

Boolean addition and multiplication satisfy the following:

1 $0 + 0 = 0$

2 $1 + 1 = 1$

3 $0 + 1 = 1 + 0 = 1$

4 $0 \times 0 = 0$

5 $0 \times 1 = 1 \times 0 = 0$

6 $1 \times 1 = 1$

7 $(0 + 1) \times 0 = 1 \times 0 = 0$ & $(0 \times 1) + 0 = 0 + 0 = 0$

8 $(0 + 1) \times 1 = 1 \times 1 = 1$ & $(0 \times 1) + 1 = 0 + 1 = 1$

9 $(0 + 1) + 0 = 0 + (1 + 0)$ & $(0 \times 1) \times 1 = 0 \times (1 \times 1)$.

Further RL_1 satisfies $0' = 1$ and hence $1' = 0$. So RL_1 is a Boolean algebra. For any elements of RL_1, $A_k + A_l$ is defined by elementwise Boolean operations of + and x. Then, any elements A_k and A_l of RL_1 satisfy the lattice conditions given below:

l_1 $A_k + A_l = A_l + A_k$ & $A_k \times A_l = A_l \times A_k$

l_2 $(A_k + A_l) + A_m = A_k + (A_l + A_m)$ &

$A_k \times A_l) \times A_m = A_k \times (A_l \times A_m)$

l_3 $A_k + A_k = A_k$ & $A_k \times A_k = A_k$

l_4 $(A_k + A_l) \times A_k = A_k$ & $(A_l \times A_m) + A_l = A_l$

Let us define $\underset{\sim}{0} = \prod_{k=1}^{K} A_k$ and $\underset{\sim}{1} = \sum_{k=1}^{K} A_k$; then the complement A_k'

is defined by $A_k + A_k' = \underset{\sim}{1}$ and $A_k \times A_k' = \underset{\sim}{0}$ with elementwise Boolean operations of + and x. The distributive laws are also satisfied from properties 7 and 8, that is $A_k \times (A_l + A_m) = A_k \times A_l + A_k \times A_m$. Therefore RL_1 becomes a Boolean algebra. In the example of our 3 x 5 incidence matrix, the elements $\underset{\sim}{0}$ and $\underset{\sim}{1}$ are given by $\underset{\sim}{0} = (0\ 0\ 0\ 0\ 1)$ and $\underset{\sim}{1} = (1\ 1\ 1\ 1\ 1)$.

Example:

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\underset{\sim}{0} = \prod_{k=1}^{K} A_k = (0\ 0\ 0\ 0\ 1)$, and $\underset{\sim}{1} = \sum_{k=1}^{K} A_k = (1\ 1\ 1\ 1\ 1)$,
$A_1' = (0\ 1\ 0\ 1\ 1)$, $A_2' = (1\ 0\ 1\ 1\ 1)$, and $A_3 = (1\ 0\ 0\ 0\ 1)$.

Several properties of RL_1 are introduced below:

Property 1 RL_1 is a subset of all possible response patterns and is closed with respect to the Boolean operations.

Property 2 If Q is the n x n identity matrix, then $RL_1 = RL$.

Property 3 If $A_k \geq A_l$ then $A_k + A_l = A_k$ and $A_k \times A_l = A_l$.

Example: Since $A_3 \geq A_2$, $A_2 + A_3 = A_3$ and $A_2 \times A_3 = A_2$.

Property 4 If $A_k \geq A_l$ then $A_k' \leq A_l'$, $(A_k + A_l)' = A_k'$ and $(A_k \times A_l)' = A_l'$.

Property 5 If Q is a Kxn lower triangle matrix (or Guttman scale matrix) then RL_1 consists of K row vectors.

If Q is a Guttman scale matrix, then the row vectors are totally ordered, $A_1 \leq A_2 \leq \ldots \leq A_K$. For any k and l, with $k \geq 1$, $A_k + A_l = A_k$ and $A_k \times A_l = A_l$ from Property 3. Moreover, the identity $\underset{\sim}{1}$ will be A_k and the null element $\underset{\sim}{0}$ will be A_1.

Incidence matrices having this form are often seen in attitude tests where measures are coded by ratings. Models such as Samejima's graded response model or Masters' partial credit model will be suitable to this form of incidence matrices. These models were developed to measure an ordered trait. For such a trait, linearly ordered levels or categories within an item exist.

As a hypothetical example, suppose there are three items:

1) Add 2/3 and 2/3, then reduce the answer to its simplest form,

2) Add 1/3 and 1/3, and

3) What is the common denominator of 1/3 and 1/5?

Then the attributes are:

A_1: Simplify to the simplest form,

A_2: Get the numerator, and

A_3: Get the denominator.

The incidence matrix is:

$$Q = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{item 1} & \text{item 2} & \text{item 3} \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \text{level 1: can do} \\ \text{level 2: can do } A_3 \text{ and } A_2 \\ \text{level 3: can do all} \end{array}$$

Thus, scores for the levels will be 1, 2, and 3, respectively.

Property 6 The complement of the sum of A_k and A_l with respect to Boolean addition is the product of the complements of A_k and A_l, $(A_k + A_l)' = A'_k \times A'_l$.

Property 7 The complement of the product of A_k and A_l with respect to Boolean product is the sum of the complements of A_k and A_l, i.e., $(A_k \times A_l)' = A_k' + A_l'$.

Definition 6 A chain is a subset of RL_1 in which all the elements are totally ordered with respect to $\geq$ or $\leq$.

Since RL_1 is a partially ordered set, (and so are L, L_1 and RL) there are usually more than one chain. The order relation is not applicable to two different elements coming from two different chains. Moreover, two chains may contain the same elements in common. Therefore, a tree graph can be drawn by connecting the elements in the chains (Tatsuoka & Tatsuoka, 1990).

The next section introduces a new function by which the universal set of ideal response patterns (or all possible states

of knowledge) is obtainable from an incidence matrix, and gives their descriptive meanings. The description of states are given by a list of combinatorial "can/cannot" attributes.

Boolean Description Function: Determination of Ideal Response Patterns As Error-Free States of Knowledge and Capabilities

There are several interesting relationships between the K-dimensional unit vector e_k and A_k.

Property 8 The unit vector e_k of the latent variable space uniquely corresponds to attribute vector A_k and the Boolean sum of e_k, $\sum_k e_k$ corresponds to the sum of A_k, $\sum_k A_k$. Similarly, the Boolean product of the elements of e_k, $\prod_k e_k$ uniquely corresponds to that of A_k, $\prod_k A_k$.

Since our goal is to draw some inferences about latent score patterns Y from observable information of item response patterns, it is necessary to introduce a series of hypotheses which convert the latent-but-interpretable information into observable-and-interpretable information. The observable information in our context is obtainable only from item responses and we do not assume observable information from the latent scores.

Definition 7 A hypothesis H_k is the statement that "one cannot do attribute A_k correctly but can do the remaining of the attributes." H_k will produce the item pattern which is the complement of A_k and represent an interpretable state of

knowledge.

It is clear that if a student cannot perform A_k correctly but can do the remaining of the attributes right, then the items involving attribute A_k will have the score of zero but the items not involving A_k will get the score of 1s. The mapping function, $A_k \rightarrow A_k'$ that takes the complement is equivalent to applying the hypothesis H_k.

Property 9 Taking Hypothesis H_k is equivalent to taking the complement of A_k and is denoted by A'_k.

Property 10 The hypothesis $H_{k1+\ldots+k1}$ is "one cannot do any of the attributes $A_{k1}, A_{k2}, \ldots, A_{k1}$ correctly but can do the rest of the attributes".

Taking the hypothesis $H_{k1+\ldots+k1}$ is equivalent to taking the complement of the addition of $A_{k1}, \ldots, A_{k1}$ i.e.,

$$(A_{k1} + \ldots + A_{k1})' = A_{k'1} \times \ldots \times A_{k'1}. \qquad (12)$$

The item pattern will be

$X_j = 0$ if $Q_{kj} = 1$ if there is at least one k in the set $(k_1, k_2, \ldots, k_1)$

$X_j = 1$ if $Q_{kj} = 0$ for all k in the set $(k_1, k_2, \ldots, k_1)$.

As an example, we use the incidence matrix of order 3 x 5 given on p. 11. Table 1 shows various hypotheses and their descriptive

outcomes and resulting ideal item patterns.

Table 1 to be inserted about here

The description of "cannot/can" in the second column of Table 1 corresponds to the latent-score patterns of y's given in Table 2. The hypothesis defined in Proerties 9 and 10 and Equation (13) provide us with a mapping between attribute patterns and ideal item patterns.

Table 2 to be inserted about here

This mapping is a Boolean function which plays the role of uncovering the contents of a black box. In our situation, latent scores on the attributes become observable via this Boolean function.

Definition 8 The mapping f from the attribute response space to the item response space is called a Boolean Description Function. The Boolean descriptive function f satisfies the following property:

Property 11 For Boolean Description Function f and e_k,

1 $f(e_k') = A_k'$

2 $f((e_1 + e_k)') = f(e_1' \times e_k') = A_1' + A_k'$

3 $f(\underline{0}) = \sum_{k=1}^{K} A_k'$

4 $f(I) = I$

Note that $\prod_{k=1}^{K} A_k = 0$, but $(\prod_{k=1}^{K} A_k)' = \sum_{k=1}^{K} A_k' \neq I$ because $\sum_{k=1}^{K} A_k = I$ but $\sum_{k=1}^{K} A_k' \neq I$ in RL_1.

Since RL_1 is a Ring, it is natural to consider the hypotheses that involve interactions of two or more attributes.

Property 12 The hypothesis $H_{k1\,x\ldots x\,k1}$ is that "one cannot do attributes $A_{k1},\ldots A_{k1}$ when all of them are involved in a single item but can do each separately and can do the remaining attributes". This hypothesis corresponds to

$$(A_{k1}\,x\ldots x\,A_{k1})' = A_{k'1} +\ldots+ A_{k'1}. \qquad (14)$$

The item pattern will be

$X_j = 0$ if $Q_{k1j} = Q_{k2j} =\ldots Q_{k1j} = 1$

$X_j = 1$ if there is at least one k_t such that $Q_{ktj} = 0$ for kt, t = 1, 2,...,1.

The hypotheses of interactions, (14) also produce the ideal item patterns that can be characterized by "can/cannot attributes".

Insert Table 3 about here

In our situation, the latent score patterns of the

attributes become observable via the Boolean description function. If the attribute response space is considered as a linear vector space of Y, then the ideal item response patterns generated by the hypothesis introduced in Property 10 will be sufficient to describe students' states of knowledge and capabilities. But as can be seen in Table 3, RL_1 contains other ideal item patterns generated by the hypotheses $H_{k1xk2x\ldots xk1}$ which involve the interaction of latent score Ys. These patterns do not correspond to the latent attribute score patterns Y in the linear vector space spanned by the e_ks. For example, the ideal item response pattern corresponding to the interaction of attribute scores $y_1 \times y_2$ is produced by $H_{y1 \times y2}$. In other words, the ideal pattern corresponding to the interaction $y_1 \times y_2$ contains 0s only for the items that involve both the attributes A_1 and A_2. Since the current test theories such as Item Response Theory models require the assumption of conditional independence of item responses, they may not be applicable to the dataset obtained through the hypothesis of the interaction of scores (y_1y_2, or X_1X_2). We will restrict the scope of this study to the linear hypothesis of $H_{k1+\ldots+k1}$, which requires only linearity of y.

The Boolean description function f is not a one-to-one function. As can be seen in Table 1, hypotheses H_3 and H_{2+3} yield the identical item pattern (1 0 0 0 0), and so do H_0 and H_{1+3}. The

ideal item patterns resulting from application of two different hypotheses may not be always different, and indeed there is a systematic relation when two hypotheses produce the same ideal item pattern. Property 3, needless to say, implies that any element A_1 smaller than A_k with respect to the order $\geq$ in L_1 "degenerates" so that addition of A_k and A_1 becomes A_k. That is, $A_1 + A_k = A_k$ if $A_k \geq A_1$. Similarly, $A_k \times A_1 = A_1$ if $A_k \geq A_1$.

A special example that is affected by this degenerative property is the incidence matrix of Guttman type. This type of incidence matrix produces K elements consisting of the original row vectors because the row vectors become a single chain of length K. The 3 x 5 incidence matrix used as example above often has two chains, A_3, and $A_1 \geq A_2$. The distinct elements will be A_3 and A_1, A_2,

$A_3 + A_1 = (1\ 1\ 1\ 1\ 1)$, $A_2 + A_1 = (1\ 1\ 1\ 0\ 1)$,

$A_3 \times A_1 = (0\ 0\ 1\ 0\ 1)$, $A_2 \times A_1 = (0\ 0\ 0\ 0\ 1)$.

Let us introduce an important definition that will be useful for determining the number of elements in RL_1.

Definition 9 An element A of L_1 is an atom if there are no elements between A and 0, or equivalently if $A \leq B$ and $A = B$, imply $B = 0$.

Property 13 Atoms in L_1 can be generated by

$A_s = \left(\bigcap_{k \in s} A_k\right) \bigcap \left(\bigcap_{k \in s} A_k\right)'$ for all possible subsets s of

$J = (1, 2, \ldots, K)$. Or, equivalently $A_s = \bigcap_{k \in s} A_k \cap \left(\bigcup_{k \in s'} A_k\right)'$.

A_ss are prospective atoms and some of them may be equal to 0. The intersection of two different atoms is 0: $A_k \cap A_l = 0$.

Property 14 Any element B of L_1 can be written as $B = \bigcup_{k \in M} A_k$ where A_k are atoms and M is an index set.

Examples of Properties 5 and 6 are illustrated with our familiar 3 x 5 Q matrix. Let us consider the index set (1, 2, 3). Its non-empty subsets are (1), (2), (3), (1, 2), (1, 3), (2, 3) and (1, 2, 3). Then

$$a_1 = A_1 \cap (A_2 \cap A_3)' = (1\ 0\ 0\ 0\ 1)$$

$$a_2 = A_2 \cap (A_1 \cap A_3)' = (0\ 0\ 0\ 0\ 1)$$

$$a_3 = A_3 \cap (A_1 \cap A_2)' = (0\ 0\ 0\ 1\ 1)$$

$$a_{12} = A_1 \cap A_2 \cap A_3' = (0\ 0\ 0\ 0\ 1)$$

$$a_{13} = A_1 \cap A_3 \cap A_2' = (0\ 0\ 1\ 0\ 1)$$

$$a_{23} = A_2 \cap A_3 \cap A_1' = (0\ 1\ 0\ 0\ 1)$$

$$a_{123} = A_1 \cap A_2 \cap A_3 = (0\ 0\ 0\ 0\ 1)$$

As can be seen in the above examples, there are four atoms a_1, a_3, a_{13}, and a_{23} while a_2, a_{12} and a_{123} are degenerated to the 0 element of L_1. The original row vectors are written as follows:

$$A_1 = a_1 + a_{13}$$

$$A_2 = a_{23}$$

$$A_3 = a_{23} + a_{13} + a_3.$$

Since every element in RL_1 is expressed by a combination of atoms,

there are $2^4 = 16$ elements in RL_1. In general, any element in RL_1 is written by a linear combination of the atoms that are linearly independent. The number of the atoms will determine the number of elements in RL. The atoms are usually not interpretable unless a test has the identity incidence matrix. The attributes in an identity incidence matrix are atoms.

Summary and Discussion

Tatsuoka (1990) discussed an incidence matrix Q that is an indication matrix of item characteristics with respect to the underlying cognitive processes which are involved in each item. These cognitive tasks are called cognitively relevant attributes in this study. An advantage of expressing the underlying item characteristics explicitly in matrix form is a tremendous benefit: First, it enables us to use a variety of scoring methods such as right or wrong, graded scores, or partial credit scores. Second, it enables us to apply powerful mathematics to investigate systematically a variety of relationships among the unobservable attributes, between the attributes and the items. Third, it enables us to help examine the structure of a test with respect to the underlying cognitive tasks.

Since a set of sets of attributes is a Boolean algebra (Boolean Algebra has been used widely in the theory of combinatorial circuits of electricity and electronics),

unobservable performances on the attributes are viewed as unobservable electric current running through various gates if they are open. An open gate corresponds to an attribute that is answered correctly, and a closed gate to wrong answers. All the gates in a circuit must be open so that the current goes through it. An item can be answered correctly if and only if all the attributes involved in the item can be answered correctly. This is an intuitive analogy between the electricity and electronics and cognitive processes of answering the items, but Boolean Algebra used for explaining various properties of electricity and combinatorial circuits can be applied to explain the underlying cognitive processes of answering the items.

The theoretical foundation of relationships between observable item response patterns and unobservable responses on the attributes which are cognitively relevant to the items is given in this study also. A newly defined Boolean descriptive function f plays the role of a link between underlying cognitive processes of test items and all the response patterns of these items. Since the model does not expect that responses on the attributes are observable, measures of performances on the attributes can not be obtained directly. However, the Boolean descriptive function converts unobservable states of knowledge and capabilities into a set of observable item patterns which are

called ideal item patterns that are free from measurement errors. The states of knowledge and capabilities are represented by a list of "can/cannot" attributes. The increase of the numbers of states is combinatorial, but Boolean algebra provides us with mathematical tools to overcome the problem of a combinatorial explosion.

Once a list of predetermined groups or states of knowledge and capabilities is determined by a software called "BUGLIB" based on this study, then the notion of "bug distribution" (Tatsuoka and Tatsuoka, 1987; Tatsuoka, 1990) and statistical pattern classification techniques (Tatsuoka, 1985; Lachenbruch, 1975) will enable us to diagnose students' states of knowledge accurately.

Finally, we conclude the study with an important implication for modern test theory. An incidence matrix implicitly indicates that the attribute scores $y=(Y_1,Y_2,\ldots,Y_K)$ satisfy local independence by a given performance level if we assume local independence at the item level. The Item Response Theory models are built upon this conditional independence of performance level theta. However, the Boolean algebra of a set of sets of response patterns is also a Ring, so it permits us to consider the states of knowledge and capabilities derived from the interaction of attribute scores. The Boolean descriptive function generates the ideal item patterns corresponding to the states determined by

using interaction of attributes. Such errors states have been observed in many studies of "bug analysis" (Brown and Burton, 1978; Tatsuoka, 1984). A new model that does not assume local independence will be needed in the future.

References

Birkoff, B., (1970). Lattice Theory. MacMillan, New York.

References

Bock, R. D. (1972). Estimating item parameters and latent ability when the responses are scored in two or more nominal categories. Psychometrika, 46, 439-459.

Masters, G. N. (1982). A Rasch Model for partial credit scoring. Psychometrika, 47, 149-174.

Samejima, F. (1969). Estimation of latent ability using a response pattern of graded scores. Psychometrika, Monograph supplement. 34(4), Part 2.

Tatsuoka, K. K. (1990). Toward an Integration of Item-Response Theory and Cognitive Error Diagnosis. In N. Fredericksens, Glaser, Lesgold and Shafto (Eds.) Diagnostic monitoring of skill and knowledge acquisition. Lawrence Erlbaum, Hillsdale, New Jersey.

Tatsuoka M. M., & Tatsuoka K.K. (1988). Rule Space. In Kotz-Johnson (Eds.) Encyclodia of Statistical Sciences, 8. John Wiley & Sons.

Tatsuoka, K. K., & Tasuoka, M. M. (1987). Bug distribution and pattern classification. Psychometrika, 52(2), 193-206.

Tatsuoka, K. K., & Tatsuoka, M. M. (1987). Item response theory, Latent Classes and Rule space. (Research Report 87-2-ONR, Final Report). Urbana, IL: University of Illinois, CERL.

Tatsuoka, K. K. (1985). A probabilistic model for diagnosing mosconceptions in the pattern classification approach. Journal of Educational Statistics. 12(1). 55-73.

Tatsuoka, K. K. (1984). Analysis of errors in fraction addition and subtraction problems. (Final Report for Grant NIE-G-0002). Urbana, IL: University of Illinois, CERL.

Tatsuoka, K. K. (1983). Rule space: An approach for dealing with misconceptions based on item response theory. Journal of Educational Measurement. 20(4), 345-354.

Acknowledgement

This research was sponsored by the Cognitive Science Program, Cognitive and Neural Science Division, Office of Naval Research, under Grant No. N00014-90-J-1307, R&T 4421559. The author is grateful to Sung-Ho Kim, Peter Pashley, Charles Lewis and Maurice Tatsuoka for valuable discussions, comments and suggestions, June Quattromani for typing the manuscript.

Table 1

Boolean Descriptive Function: Case of Linear Hypothesis

Hypothesis	Interpretation	Ideal Response Pattern
H_0	can do everything	(1 1 1 1 1)
H_1	cannot A_1, can A_2, A_3	(0 1 0 1 0)
H_2	cannot A_2, can A_1, A_3	(1 0 1 1 0)
H_3	cannot A_3, can A_1, A_2	(1 0 0 0 0)
H_{1+2}	cannot A_1, A_2, can A_3	(0 0 0 1 0)
H_{1+3}	cannot A_1, A_3, can A_2	(0 0 0 0 0)
H_{2+3}	cannot A_2, A_3, can A_1	(1 0 0 0 0)
H_{1+2+3}	cannot A_1, A_2, and A_3	(0 0 0 0 0)

Table 2

Correspondence Between Latent Attribute Space and Item Space

Hypothesis	Attribute Score	Item Score
H_0	(1 1 1)	(1 1 1 1 1)
H_1	(0 1 1)	(0 1 0 1 0)
H_2	(1 0 1)	(1 0 1 1 0)
H_3	(1 1 0)	(1 0 0 0 0)
H_{1+2}	(0 0 1)	(0 0 0 1 0)
H_{1+3}	(0 1 0)	(0 0 0 0 0)
H_{2+3}	(1 0 0)	(1 0 0 0 0)
H_{1+2+3}	(0 0 0)	(0 0 0 0 0)

Table 3

Boolean Descriptive Function: Case of Interaction

Hypothesis	Interpretation	Ideal Response Pattern
H_{2x3}	Cannot A_2 and A_3 together, can A_1	(1 0 1 1 0)
H_{1x3}	Cannot A_1 and A_3 together, can A_2	(1 1 0 1 0)

Dr. Terry Ackerman
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. James Algina
1403 Norman Hall
University of Florida
Gainesville, FL 32605

Dr. Nancy Allen
Educational Testing Service
Princeton, NJ 08541

Dr. Erling B. Andersen
Department of Statistics
Studiestraede 6
1455 Copenhagen
DENMARK

Dr. Ronald Armstrong
Rutgers University
Graduate School of Management
Newark, NJ 07102

Dr. Eva L. Baker
UCLA Center for the Study
of Evaluation
145 Moore Hall
University of California
Los Angeles, CA 90024

Dr. Laura L. Barnes
College of Education
University of Toledo
2801 W. Bancroft Street
Toledo, OH 43606

Dr. William M. Bart
University of Minnesota
Dept. of Educ. Psychology
330 Burton Hall
178 Pillsbury Dr., S.E.
Minneapolis, MN 55455

Dr. Isaac Bejar
Law School Admissions
Services
P.O. Box 40
Newtown, PA 18940-0040

Dr. Anne Belsnd
Educational Testing Service
Princeton, NJ 08541

Dr. Ira Bernstein
Department of Psychology
University of Texas
P.O. Box 19528
Arlington, TX 76019-0528

Dr. Menucha Birenbaum
School of Education
Tel Aviv University
Ramat Aviv 69978
ISRAEL

Dr. Bruce Bloxom
Defense Manpower Data Center
99 Pacific St.
Suite 155A
Monterey, CA 93943-3231

Cdt. Arnold Bohrer
Sectie Psychologisch Onderzoek
Rekruterings-En Selectiecentrum
Kwartier Koningen Astrid
Bruijnstraat
1120 Brussels, BELGIUM

Dr. Gwyneth Boodoo
Educational Testing Service
Princeton, NJ 08541

Dr. Robert Breaux
Code 252
Naval Training Systems Center
Orlando, FL 32826-3224

Dr. Robert Brennan
American College Testing
Programs
P. O. Box 168
Iowa City, IA 52243

Dr. David V. Budescu
Department of Psychology
University of Haifa
Mount Carmel, Haifa 31999
ISREAL

Dr. Gregory Candell
CTB/McGraw-Hill
2500 Garden Road
Monterey, CA 93940

Dr. John B. Carroll
409 Elliott Rd., North
Chapel Hill, NC 27514

Dr. John M. Carroll
IBM Watson Research Center
User Interface Institute, H1-B52
P.O. Box 704
Yorktown Heights, NY 10598

Dr. Robert M. Carroll
Chief of Naval Operations
OP-01B2
Washington, DC 20350

Dr. W. Chambers
Technology Manager, Code 2B
Naval Training Systems Center
12350 Research Parkway
Orlando, FL 32826-3224

Mr. Hua Hua Chang
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright St.
Champaign, IL 61820

Dr. Raymond E. Christal
UES LAMP Science Advisor
AFHRL/MOEL
Brooks AFB, TX 78235

Dr. Norman Cliff
Department of Psychology
Univ. of So. California
Los Angeles, CA 90089-1061

Director, Manpower Program
Center for Naval Analyses
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Director,
Manpower Support and
Readiness Program
Center for Naval Analysis
4401 Ford Avenue
Alexandria, VA 22302-0268

Dr. Stanley Collyer
Office of Naval Technology
Code 222
800 N. Quincy Street
Arlington, VA 22217-5000

Dr. Hans F. Crombag
Faculty of Law
University of Limburg
P.O. Box 616
Maastricht
The NETHERLANDS 6200 MD

Ms. Carolyn R. Crone
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Dr. Timothy Davey
American College Testing Program
P.O. Box 168
Iowa City, IA 52243

Dr. C. M. Dayton
Department of Measurement
Statistics & Evaluation
College of Education
University of Maryland
College Park, MD 20742

Dr. Ralph J. DeAyala
Measurement, Statistics,
and Evaluation
Benjamin Bldg., Rm. 4112
University of Maryland
College Park, MD 20742

Dr. Lou DiBello
CERL
University of Illinois
103 South Mathews Avenue
Urbana, IL 61801

Dr. Dattprasad Divgi
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. Neil Dorans
Educational Testing Service
Princeton, NJ 08541

Dr. Fritz Drasgow
University of Illinois
Department of Psychology
603 E. Daniel St.
Champaign, IL 61820

Defense Technical
Information Center
Cameron Station, Bldg 5
Alexandria, VA 22314
(2 Copies)

Dr. Stephen Dunbar
224B Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. James A. Earles
Air Force Human Resources Lab
Brooks AFB, TX 78235

Dr. Susan Embretson
University of Kansas
Psychology Department
426 Fraser
Lawrence, KS 66045

Dr. George Englehard, Jr.
Division of Educational Studies
Emory University
210 Fishburne Bldg.
Atlanta, GA 30322

ERIC Facility-Acquisitions
2440 Research Blvd, Suite 550
Rockville, MD 20850-3238

BEST COPY AVAILABLE

Dr. Benjamin A. Fairbank
Operational Technologies Corp.
5825 Callaghan, Suite 225
San Antonio, TX 78228

Dr. Marshall J. Farr, Consultant
Cognitive & Instructional Sciences
2520 North Vernon Street
Arlington, VA 22207

Dr. P-A. Federico
Code 51
NPRDC
San Diego, CA 92152-6800

Dr. Leonard Feldt
Lindquist Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. Richard L. Ferguson
American College Testing
P.O. Box 168
Iowa City, IA 52243

Dr. Gerhard Fischer
Liebiggasse 5/3
A 1010 Vienna
AUSTRIA

Dr. Myron Fischl
U.S. Army Headquarters
DAPE-MRR
The Pentagon
Washington, DC 20310-0300

Prof. Donald Fitzgerald
University of New England
Department of Psychology
Armidale, New South Wales 2351
AUSTRALIA

Mr. Paul Foley
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Alfred R. Fregly
AFOSR/NL, Bldg. 410
Bolling AFB, DC 20332-6448

Dr. Alice Gerb
Educational Testing Service
Princeton, NJ 08541

Dr. Robert D. Gibbons
Illinois State Psychiatric Inst.
Rm 529W
1601 W. Taylor Street
Chicago, IL 60612

Dr. Janice Gifford
University of Massachusetts
School of Education
Amherst, MA 01003

Dr. Drew Gitomer
Educational Testing Service
Princeton, NJ 08541

Dr. Robert Glaser
Learning Research
& Development Center
University of Pittsburgh
3939 O'Hara Street
Pittsburgh, PA 15260

Dr. Karen Gold
Educational Testing Service
Princeton, NJ 08541

Dr. Timothy Goldsmith
Department of Psychology
University of New Mexico
Albuquerque, NM 87131

Dr. Sherrie Gott
AFHRL/MOMJ
Brooks AFB, TX 78235-5601

Dr. Bert Green
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Michael Habon
DORNIER GMBH
P.O. Box 1420
D-7990 Friedrichshafen 1
WEST GERMANY

Prof. Edward Haertel
School of Education
Stanford University
Stanford, CA 94305

Dr. Ronald K. Hambleton
University of Massachusetts
Laboratory of Psychometric
and Evaluative Research
Hills South, Room 152
Amherst, MA 01003

Dr. Delwyn Harnisch
University of Illinois
51 Gerty Drive
Champaign, IL 61820

Dr. Grant Henning
Mail Stop 18-P
Educational Testing Service
Princeton, NJ 08541

Ms. Rebecca Hetter
Navy Personnel R&D Center
Code 63
San Diego, CA 92152-6800

Dr. Thomas M. Hirsch
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. Paul W. Holland
Educational Testing Service, 21-T
Rosedale Road
Princeton, NJ 08541

Dr. Paul Horst
677 G Street, #184
Chula Vista, CA 92010

Ms. Julia S. Hough
Cambridge University Press
40 West 20th Street
New York, NY 10011

Dr. William Howell
Chief Scientist
AFHRL/CA
Brooks AFB, TX 78235-5601

Dr. Lloyd Humphreys
University of Illinois
Department of Psychology
603 East Daniel Street
Champaign, IL 61820

Dr. Steven Hunka
3-104 Educ. N.
University of Alberta
Edmonton, Alberta
CANADA T6G 2G5

Dr. Huynh Huynh
College of Education
Univ. of South Carolina
Columbia, SC 29208

Dr. Martin J. Ippel
Postbus 9555
2300 RB Leiden
THE NETHERLANDS

Dr. Robert Jannarone
Elec. and Computer Eng. Dept.
University of South Carolina
Columbia, SC 29208

Dr. Kumar Joag-dev
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright Street
Champaign, IL 61820

Dr. Peder Johnson
Department of Psychology
University of New Mexico
Albuquerque, NM 87131

Dr. Douglas H. Jones
1280 Woodfern Court
Toms River, NJ 08753

Dr. Brian Junker
Carnegie-Mellon University
Department of Statistics
Schenley Park
Pittsburgh, PA 15213

Dr. Michael Kaplan
Office of Basic Research
U.S. Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333-5600

Dr. Milton S. Katz
European Science Coordination
Office
U.S. Army Research Institute
Box 65
FPO New York 09510-1500

Prof. John A. Keats
Department of Psychology
University of Newcastle
N.S.W. 2308
AUSTRALIA

Mr. Hae-Rim Kim
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright St.
Champaign, IL 61820

Dr. Jwa-keun Kim
Department of Psychology
Middle Tennessee State
University
P.O. Box 522
Murfreesboro, TN 37132

Dr. Sung-Ho Kim
Educational Testing Service
Princeton, NJ 08541

Dr. G. Gage Kingsbury
Portland Public Schools
Research and Evaluation Department
501 North Dixon Street
P. O. Box 3107
Portland, OR 97209-3107

Dr. William Koch
Box 7246, Meas. and Eval. Ctr.
University of Texas-Austin
Austin, TX 78703

Dr. Richard J. Koubek
School of Civil Engineering
Grissom Hall
Purdue University
West Lafayette, IN 47907

Dr. Leonard Kroeker
Navy Personnel R&D Center
Code 62
San Diego, CA 92152-6800

Dr. Jerry Lehnus
Defense Manpower Data Center
Suite 400
1600 Wilson Blvd
Rosslyn, VA 22209

Dr. Thomas Leonard
University of Wisconsin
Department of Statistics
1210 West Dayton Street
Madison, WI 53705

Dr. Richard Lesh
Educational Testing Service
Princeton, NJ 08541

Dr. Michael Levine
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Charles Lewis
Educational Testing Service
Princeton, NJ 08541-0001

Ms. Hsin-hung Li
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright St.
Champaign, IL 61820

Mr. Rodney Lim
University of Illinois
Department of Psychology
603 E. Daniel St.
Champaign, IL 61820

Dr. Robert L. Linn
Campus Box 249
University of Colorado
Boulder, CO 80309-0249

Dr. Robert Lockman
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. Frederic M. Lord
Educational Testing Service
Princeton, NJ 08541

Dr. Richard Luecht
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. George B. Macready
Department of Measurement
Statistics & Evaluation
College of Education
University of Maryland
College Park, MD 20742

Dr. Gary Marco
Stop 31-E
Educational Testing Service
Princeton, NJ 08451

Dr. Clessen J. Martin
Office of Chief of Naval
Operations (OP 13 F)
Navy Annex, Room 2832
Washington, DC 20350

Dr. Shin-ichi Mayekawa
The National Center for University
Entrance Examinations
2-19-23 KOMABA, MEGURO-KU
Tokyo 153
JAPAN

Dr. James R. McBride
HumRRO
6430 Elmhurst Drive
San Diego, CA 92120

Dr. Clarence C. McCormick
HQ, USMEPCOM/MEPCT
2500 Green Bay Road
North Chicago, IL 60064

Mr. Christopher McCusker
University of Illinois
Department of Psychology
603 E. Daniel St.
Champaign, IL 61820

Dr. Robert McKinley
Educational Testing Service
Princeton, NJ 08541

Dr. Michael McNeese
DET-1, AL/MED
BLDG 248
Wright-Patterson AFB, OH 45432

Mr. Alan Mead
c/o Dr. Michael Levine
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Timothy Miller
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. Robert Mislevy
Educational Testing Service
Princeton, NJ 08541

Dr. William Montague
NPRDC Code 13
San Diego, CA 92152-6800

Ms. Kathleen Moreno
Navy Personnel R&D Center
Code 62
San Diego, CA 92152-6800

Headquarters Marine Corps
Code MPI-20
Washington, DC 20380

Dr. Ratna Nandakumar
Educational Studies
Willard Hall, Room 213E
University of Delaware
Newark, DE 19716

Library, NPRDC
Code P201L
San Diego, CA 92152-6800

Librarian
Naval Center for Applied Research
in Artificial Intelligence
Naval Research Laboratory
Code 5510
Washington, DC 20375-5000

Dr. Harold F. O'Neil, Jr.
School of Education - WPH 801
Department of Educational
Psychology & Technology
University of Southern California
Los Angeles, CA 90089-0031

Dr. James B. Olsen
WICAT Systems
1875 South State Street
Orem, UT 84058

Office of Naval Research,
Code 1142CS
800 N. Quincy Street
Arlington, VA 22217-5000
(6 Copies)

Dr. Judith Orasanu
Basic Research Office
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Dr. Jesse Orlansky
Institute for Defense Analyses
1801 N. Beauregard St.
Alexandria, VA 22311

Dr. Peter J. Pashley
Educational Testing Service
Rosedale Road
Princeton, NJ 08541

Wayne M. Patience
American Council on Education
GED Testing Service, Suite 20
One Dupont Circle, NW
Washington, DC 20036

Dr. James Paulson
Department of Psychology
Portland State University
P.O. Box 751
Portland, OR 97207

Dept. of Administrative Sciences
Code 54
Naval Postgraduate School
Monterey, CA 93943-5026

Dr. Mark D. Reckase
ACT
P. O. Box 168
Iowa City, IA 52243

Dr. Malcolm Ree
AFHRL/MOA
Brooks AFB, TX 78235

Mr. Steve Reiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455-0344

Dr. W. A. Rizzo
Head, Human Factors Division
Naval Training Systems Center
Code 26
12350 Research Parkway
Orlando, FL 32826-3224

Dr. Carl Ross
CNET-PDCD
Building 90
Great Lakes NTC, IL 60088

Mr. Louis Roussos
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright St.
Champaign, IL 61820

Dr. J. Ryan
Department of Education
University of South Carolina
Columbia, SC 29208

BEST COPY AVAILABLE

Dr. Fumiko Samejima
Department of Psychology
University of Tennessee
310B Austin Peay Bldg.
Knoxville, TN 37916-0900

Mr. Drew Sands
NPRDC Code 62
San Diego, CA 92152-6800

Mr. Kenneth Sarno
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Janice Scheuneman
Educational Testing Service
Princeton, NJ 08541

Lowell Schoer
Psychological & Quantitative Foundations
College of Education
University of Iowa
Iowa City, IA 52242

Dr. Mary Schratz
4100 Parkside
Carlsbad, CA 92008

Dr. Dan Segall
Navy Personnel R&D Center
San Diego, CA 92152

Mr. Robert Semmes
N218 Elliott Hall
Department of Psychology
University of Minnesota
Minneapolis, MN 55455

Dr. Robin Shealy
Illinois State Water Survey
Room 149
2204 Griffith Dr.
Champaign, IL 61820

Ms. Kathleen Sheehan
Educational Testing Service
Princeton, NJ 08541

Dr. Kazuo Shigemasu
7-9-24 Kugenuma-Kaigan
Fujisawa 251
JAPAN

Dr. Randall Shumaker
Naval Research Laboratory
Code 5510
4555 Overlook Avenue, S.W.
Washington, DC 20375-5000

Dr. Richard E. Snow
School of Education
Stanford University
Stanford, CA 94305

Dr. Richard C. Sorensen
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Judy Spray
ACT
P.O. Box 168
Iowa City, IA 52243

Dr. Martha Stocking
Educational Testing Service
Princeton, NJ 08541

Dr. Peter Stoloff
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. William Stout
University of Illinois
Department of Statistics
101 Illini Hall
725 South Wright St.
Champaign, IL 61820

Dr. Hariharan Swaminathan
Laboratory of Psychometric and Evaluation Research
School of Education
University of Massachusetts
Amherst, MA 01003

Mr. Brad Sympson
Navy Personnel R&D Center
Code 62
San Diego, CA 92152-6800

Dr. John Tangney
AFOSR/NL, Bldg. 410
Bolling AFB, DC 20332-6448

Dr. Kikumi Tatsuoka
Educational Testing Service
Mail Stop 03-T
Princeton, NJ 08541

Dr. Maurice Tatsuoka
Educational Testing Service
Mail Stop 03-T
Princeton, NJ 08541

Dr. David Thissen
Department of Psychology
University of Kansas
Lawrence, KS 66044

Mr. Thomas J. Thomas
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Mr. Gary Thomasson
University of Illinois
Educational Psychology
Champaign, IL 61820

Mr. Sherman Tsien
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Robert Tsutakawa
University of Missouri
Department of Statistics
222 Math. Sciences Bldg.
Columbia, MO 65211

Dr. Ledyard Tucker
University of Illinois
Department of Psychology
603 E. Daniel Street
Champaign, IL 61820

Dr. David Vale
Assessment Systems Corp.
2233 University Avenue
Suite 440
St. Paul, MN 55114

Dr. Frank L. Vicino
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Howard Wainer
Educational Testing Service
Princeton, NJ 08541

Dr. Michael T. Waller
University of Wisconsin-Milwaukee
Educational Psychology Department
Box 413
Milwaukee, WI 53201

Dr. Ming-Mei Wang
Educational Testing Service
Mail Stop 03-T
Princeton, NJ 08541

Dr. Thomas A. Warm
FAA Academy AAC934D
P.O. Box 25082
Oklahoma City, OK 73125

Dr. Brian Waters
HumRRO
1100 S. Washington
Alexandria, VA 22314

Dr. David J. Weiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455-0344

Dr. Ronald A. Weitzman
Box 146
Carmel, CA 93921

Major John Welsh
AFHRL/MOAN
Brooks AFB, TX 78223

Dr. Douglas Wetzel
Code 51
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Rand R. Wilcox
University of Southern California
Department of Psychology
Los Angeles, CA 90089-1061

German Military Representative
ATTN: Wolfgang Wildgrube
Streitkraefteamt
D-5300 Bonn 2
4000 Brandywine Street, NW
Washington, DC 20016

Dr. David Wiley
School of Education
Northwestern University
Evanston, IL 60201

Dr. Charles Wilkins
Navy Personnel R&D Center
Code 13
San Diego, CA 92152

Dr. Bruce Williams
Department of Educational Psychology
University of Illinois
Urbana, IL 61801

Dr. Mark Wilson
School of Education
University of California
Berkeley, CA 94720

Dr. Hilda Wing
Federal Aviation Administration
800 Independence Ave, SW
Washington, DC 20591

Mr. John H. Wolfe
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. George Wong
Biostatistics Laboratory
Memorial Sloan-Kettering Cancer Center
1275 York Avenue
New York, NY 10021

BEST COPY AVAILABLE

Dr. Wallace Wulfeck, III
Science Advisor
NAVOP 01SA/PERS 00R
Washington, DC 20350

Dr. Kentaro Yamamoto
02-T
Educational Testing Service
Rosedale Road
Princeton, NJ 08541

Ms. Duanli Yan
Educational Testing Service
Princeton, NJ 08541

Dr. Wendy Yen
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

Dr. Joseph L. Young
National Science Foundation
Room 320
1800 G Street, N.W.
Washington, DC 20550

Mr. Anthony R. Zara
National Council of State
Boards of Nursing, Inc.
625 North Michigan Avenue
Suite 1544
Chicago, IL 60611